Variation of Properties during a Vessel Discharge

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Abstract

This paper explicates a study of a pressurized flow when a closed tank is suddenly opened and the flow passes through an orifice until the tank is at equilibrium with the surroundings. The research analyzes the evolution of this adiabatic flow over time considering its three most important regimes: a choked condition, an unchoked condition, and an incompressible condition. In order to fully characterize the flow, the study analyzes the pressure, the temperature, the density, and the mass flow rate during the discharge of the vessel. All of these properties are studied using their own equations in their dimensionless form and later plotted using the math software program MATLAB. Given some inputs such as the initial and final dimensionless pressures and the specific heat ratio, it is possible to determine the dimensionless time until the tank is drained and the evolutions of the properties.

Nomenclature

γ	=	specific heat ratio [dimensionless]
A_0	=	area of the orifice [m^2]
C_c	=	coefficient of contraction [dimensionless]
C_d	=	coefficient of discharge [adim]
K_f	=	flow coefficient [dimensionless]
M	=	mach number [dimensionless]
$\dot{m}$	=	mass flow rate [kg/s]
P_i	=	initial pressure of the tank [Pa]
P_f	=	pressure outside the tank [Pa]
ρ_i	=	initial density of the air in the tank [kg/m^3]
T_i	=	initial temperature of the tank [K°]
V	=	volume of the tank, constant [m^3]

Introduction

Let us consider a situation in which there is a closed tank with a volume, V, full of some unspecified fluid. The fluid has some initial conditions such as T_i, ρ_i, mass, and as it is an over pressurized fluid, $P_i > P_f$. Suddenly the tank is opened, and the fluid starts to go out from the vessel through an orifice of area, A_0. Based on intuition, it can be expected that the pressure, the density, and the mass flow rate would decrease, and the temperature will also decrease, if it is an adiabatic case, or instead will be constant, if it is an isothermal case. Due to the over pressurized condition and based upon the literature of fluid mechanics, during the discharge of the vessel, the flow will behave in three different modes. At first it will be a choked flow, but as the pressure keeps dropping, the flow will become unchoked. In that moment the pressure will still be high enough to cause compressible effects on the flow, so it will be in an unchoked

compressible condition, and finally the third regime will show up when the flow becomes unchoked and incompressible.

In the paper "Experiments to Study the Gaseous Discharge and Filling of Vessels," J.Craig Dutton and Robert E. Coverdill [1] have demonstrated that in a fast-draining situation the flow is going to perform adiabatically. Based on their experimental results, it has been assumed that there is an adiabatic flow instead of isentropic, and consequently adiabatic equations have been used for determining the evolution along the three regimes.

Another fact is that the proposed problem has been analyzed for different studies, but each of them concentrates the chief issue of the study on the evolution of one property or the study of just one phase of the flow [2]. Instead, this paper collects the evolution of main properties during the three regimes and uses the specific equations that describe each of these three conditions.

First of all, this paper describes the assumptions taken and defines the dimensionless variables that would work within a general case. Then a mathematical development for finding the general equation of the flow is performed, which permits the finding of the flow equations depending-on time of each case. Finally, once the equations are derived, it is possible to plot them using MATLAB and analyze the results.

Assumptions

An effort has been made to keep the study general, but some assumptions have been taken in order to simplify the mathematical development:

1. Quasi-steady flow
2. One-dimensional flow
3. The velocity of the fluid inside the tank is zero
4. Gravitational potential energy is neglected: $gz = 0$
5. Shear or shaft work for the control volume is neglected: $\dot{W} = 0$
6. Fluid is thermally and calorically perfect: $P = \rho R T$, C_v and C_p constant
7. Adiabatic flow: no heat transfer as it is a rapid discharge, $\dot{Q} = 0$
8. Rigid tank

Dimensionless variables

Defining dimensionless variables:

$$P^* = \frac{P}{P_i}; \qquad P_f^* = \frac{P_f}{P_i}; \qquad \rho^* = \frac{\rho}{\rho_i}; \qquad T^* = \frac{T}{T_i}$$

$$m^* = \frac{\dot{m}\sqrt{RT_0}}{P_0 A}; \qquad t^* = \frac{t}{t_{char}}; \qquad t_{char} = \frac{V}{PA}; \qquad C_c = \frac{A}{A_0}$$

Mathematical procedure

To describe a flow we need the governing equations, but in this case only two are necessary: the continuity and energy conservation equations. Once these equations

are found, it is possible to study the thermodynamic relationships of the fluid to know how the fluid inside the tank expands as it is being drained.

Continuity equation

$$\frac{\partial}{\partial t}\int_V \rho \, dV + \int_S \rho V \cdot dS = 0 \tag{1}$$

Under the quasi-steady flow assumption (assumption number 1), the density is uniform throughout the control volume and can be pulled out of the integral. Then, as it is a rigid tank (assumption number 8), it is possible to take the volume out from the time derivative, leaving the density which is a function of time. Knowing the definition of mass flow rate, $\dot{m} = \rho V A$, the continuity equation becomes:

$$\frac{d\rho}{dt} + \frac{\dot{m}}{V} = 0 \tag{2}$$

Energy equation

$$\dot{Q} - \dot{W} = \frac{\partial}{\partial t}\int_V e\rho \, dV + \int_S (e + pv)\rho V \cdot dS \tag{3}$$

The first and the second term become 0 because of assumptions 7 and 5 respectively. Defining the internal energy as $e = u + \frac{v^2}{2} + gz$, substituting it in the third and fourth terms, and simplifying while making assumptions 2 and 4, an expression of continuity depending on the enthalpy is developed. Resolving the integrals it is possible to obtain a final equation:

$$\frac{d}{dt}(\rho u) + \frac{\dot{m}}{V}H = 0 \tag{4}$$

Thermodynamic relation

Combining the equations of continuity and energy in order to remove the mass flow rate and the volume as variables, another equation is obtained. Invoking assumption 7 for an adiabatic flow, the stagnation enthalpy (H) is the enthalpy of the fluid in the tank (h).

$$\frac{d}{dt}(\rho u) - \frac{d\rho}{dt}h = 0 \tag{5}$$

Following this simplification, assumption 6 implies some definitions that are helpful, such as the specific heat ratio (γ) for these type of flows.

$$u = C_v T \qquad\qquad h = C_p T \qquad\qquad \gamma = \frac{C_p}{C_v}$$

Substituting these definitions in the equation (5), continuing with the development of it, and introducing the dimensionless variables result in the isentropic relations. That means that when the tank is draining the fluid expands isentropically and the temperature and density can be expressed as a function of pressure as it follows.

$$T^* = (P^*)^{\frac{(\gamma-1)}{\gamma}} \tag{6}$$

$$\rho^* = (P^*)^{\frac{1}{\gamma}}$$

(7)

Flow equations depending on time

Due to the high difference of pressures between inside and outside of the tank, the flow will evolve through three different behaviors. These three cases have been studied, and their dimensionless time-dependent equations derived.

Choked flow equations

In this case the mass flow rate is expressed by Fliegner's formula [3] which is independent of pressure and dependent on the specific heat ratio of the fluid (γ). So, as long as the flow is choked the mass flow rate is a constant $K = 0.6847$ (taking $\gamma = 1.4$) [4].

(8)

$$m^* = \sqrt{\gamma}\left[\frac{\gamma+1}{2}\right]^{\frac{-(\gamma+1)}{2(\gamma-1)}}$$

By introducing the dimensional form of the mass flow rate to the continuity equation (2), non-dimensionalizing all the variables, and defining the characteristic time, it is possible to reach a simple expression with all the variables of interest.

(9)

$$\frac{d\rho^*}{dt^*} + \frac{K}{\sqrt{\gamma}}\frac{P^*}{\sqrt{T^*}} = 0$$

When the variables have been substituted using the isentropic relations and the equation depends only on time and pressure, it is possible to integrate it and arrive at an expression of the pressure depending on the time. Using the isentropic relations again results in temperature and density expressions.

(10)

$$P^* = \left[1 + \left(\frac{\gamma-1}{2}\right)\left(\frac{\gamma+1}{2}\right)^{\frac{-(\gamma+1)}{2(\gamma-1)}} t^*\right]^{\frac{-2\gamma}{(\gamma-1)}}$$

(11)

$$\rho^* = (P^*)^{\frac{1}{\gamma}}$$

(12)

$$T^* = (P^*)^{\frac{\gamma-1}{\gamma}}$$

Unchoked compressible flow equation

For determining when the flow becomes unchoked, we used the critical pressure ratio condition.

(13)

$$\frac{P_f^*}{P^*} = \left(\frac{2}{\gamma+1}\right)^{\frac{\gamma}{\gamma-1}}$$

This expression depends solely on the heat specific ratio, so it is a constant for each value of γ. When the pressure ratio is greater than that constant, it means that the flow is still choked, and when it becomes lower, it means that the flow is unchoked. Taking the value $= 1.4$, the constant is 0.528.

The procedure for finding the equations of the flow is the same as before, but the expression for the mass flow rate becomes dependent on the final pressure. Once the

expression for mass flow has been introduced into the continuity expression and it has been non-dimensionalised, then it is possible to integrate. An initial condition is taken as the "unchoked time," which is the moment when the flow passes from the choked condition to the unchoked. Then, as a final condition of the integration, an arbitrary time is used. Doing the integration results in an expression of the time as a function of pressure.

$$m^* = \left(\frac{2\gamma}{\gamma-1}\right)^{\frac{1}{2}}\left[1-\left(\frac{P_f^*}{P^*}\right)^{\frac{\gamma-1}{\gamma}}\right]^{\frac{1}{2}}\left(\frac{P_f^*}{P^*}\right)^{\frac{1}{\gamma}}$$

$$t^* = t_{unch}^* + \left(\frac{2}{\gamma-1}\right)^{\frac{1}{2}}C_c(P_f^*)^{\frac{-(\gamma-1)}{2\gamma}}\left[\left(\frac{x^3}{4}+\frac{5}{8}x\right)(x^2+1)^{\frac{1}{2}}+\frac{3}{8}\ln\left(x+(x^2+1)^{\frac{1}{2}}\right)\right]_x^{x_{unch}}$$

$$x = \left[\left(\frac{P^*}{P_f^*}\right)^{\frac{\gamma-1}{\gamma}}-1\right]^{\frac{1}{2}}$$

$$\rho^* = (P^*)^{\frac{1}{\gamma}}$$

$$T^* = (P^*)^{\frac{\gamma-1}{\gamma}}$$

In this equation of time-pressure, there is a new parameter called the coefficient of contraction, C_c, due to the vena contracta effect. This coefficient appears because the discharge of the vessel is through an orifice, and when this happens the flow acts as shown in the figure below and contracts itself. The coefficient is the relation between the areas of the orifice and the point where the flow is most compressed. The values for this parameter are between 0.61-0.64.

$$C_c = \frac{A}{A_o}$$
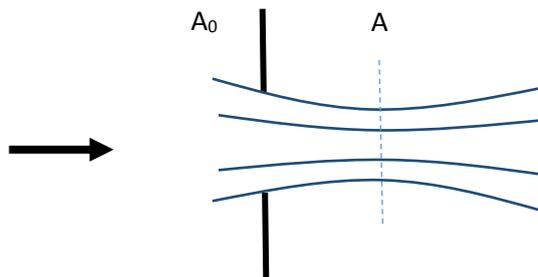
1. Vena contracta effect

Unchoked incompressible flow equations

It is considered that when Mach number is 0.3 or less the fluid is incompressible, and this is the condition used for determining the change of the regime from compressible to incompressible. Knowing the pressures and the specific heat ratio, it is possible to calculate the Mach number as follows [5]:

$$M = \left[\frac{2}{\gamma - 1} \left\{ \left(\frac{P^*}{P_f^*} \right)^{\frac{\gamma - 1}{\gamma}} - 1 \right\} \right]^{\frac{1}{2}}$$

When the incompressible condition is reached, it is assumed that there is quasi-steady flow. It is also assumed that there is no shaft work as there is no turbine or pump. Moreover, as the discharge is through an orifice, the friction losses are also neglected. Adding the assumption of inviscid and laminar flow, it is possible to use the Bernoulli equation to characterize this flow.

The flow coefficient K_f is a parameter that represents the relation between the pressure drops and the mass flow rate. The flow coefficient is the product between the coefficient of discharge, C_d, and the velocity approach factor, $\frac{1}{\sqrt{1 - \beta^4}}$.

- The coefficient of discharge is an experimental non-dimensional number which is used for calculating the mass flow rate while a tank discharges to the environment. It depends on the shape of the orifice and its area, but for this study we've used the average number of 0.68.

- The velocity approach factor which relates the velocity at the orifice to the velocity at the point of maximum contraction of the flow. In its expression there is a new parameter β which is a relation between diameters of the two points of interest.

As was said before to find the governing equations of the flow, the Bernoulli equation and the continuity equation were used. Introducing the parameters for the loss effect, it is possible to achieve the mass flow rate equation.

$$m^* = K_f C_c \sqrt{2 \left(P^* - P_f^* \right)} \tag{17}$$

Then from the ideal gas equation $PV = mRT$ we can develop the time-dependent equations, just like the other two cases.

$$t^* = t_{inc} - \frac{\sqrt{2\gamma}}{C_c K_f} \frac{1}{T^*} \left(\sqrt{P_{inc}^* - P_f^*} - \sqrt{P^* - P_f^*} \right) \tag{18}$$

$$\rho^* = cnt$$

$$T^* = (P^*)^{\frac{\gamma - 1}{\gamma}}$$

Results

Using the math program MATLAB with all the flow equations the following figure was created:

PROPERTIES VS TIME (ADIABATIC)
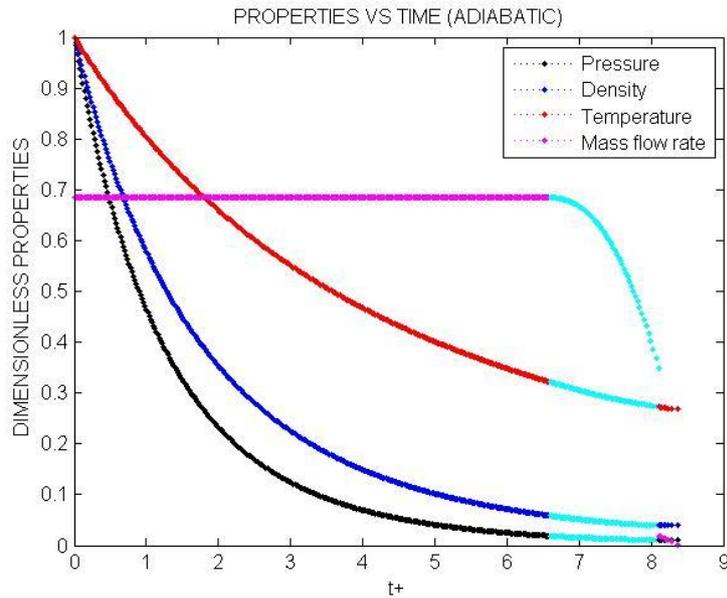
Every color represents one of the studied variables (see legend). The cyan color represents the variation of the properties during the unchoked compressible regime, so where the color changes it signifies a change of a regime. When $t^* = 6.58$ the flow changes its condition from the choked regime to the unchoked compressible regime, and when $t^* = 8.069$ the flow changes its condition again from the unchoked compressible regime to unchoked incompressible regime.

Other studies can be carried out with these equations to study the effect of the final pressure, what happens when the difference between pressures is small, or to study the times of discharge for other types of orifices. As can be seen in the next two graphs, these parameters change the discharge process. The graph on the left is using a smaller pressure diference so the discharge is faster, less than 4 seconds. The graph on the right is using $C_c = 1$, which affects the draining time.

PROPERTIES VS TIME (ADIABATIC)
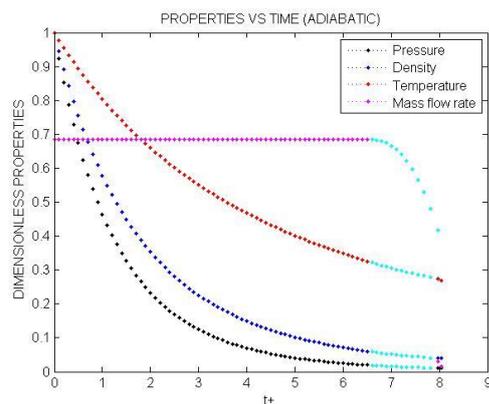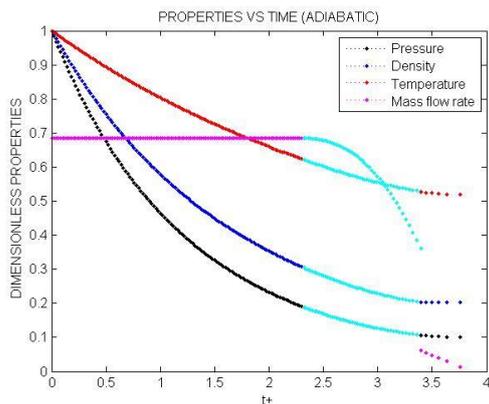
Conclusions

This paper has developed the mathematical procedure for obtaining the equations of flow in each of the three regimes of a high pressure tank being drained. The graphs show the feasibility of using compressible and incompressible flow calculations to determine a generalized time to drain a tank. Furthermore, from the

graphics it can be seen that the main regime condition is choked flow because of the difference imposed between the initial and final dimensionless pressures. If the difference is smaller, the choked regime condition is reduced because there is not enough pressure to maintain that condition. Also it can be seen that the mass flow rate has a discontinuity between the compressible and incompressible regimes due to the fast discharge of the vessel. When the flow reaches the incompressible condition the tank is almost empty.

The completion of the equations and the graphics are very important in understanding how the basis of rocket propulsion works. Because of Newton's third law, the law of action and reaction, when the deposit of the rocket is draining, the fluid that is going out will produce thrust. So, knowing the mass flow rate at each moment can be used to calculate how much thrust is going to be. Also, the period of time when the rocket is going to have maximum thrust can be determined, which is going to be during the compressible regime because it is when there is maximum flow rate. Furthermore, as the development has been done for a generalized fluid, the equations can be applied to any kind of rocket propulsion, such as water, air, or any other fluid.

References

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[4] O'Keefe Controls Co., *Choked Flow of Gases*, Tutorial, Connecticut (2003).

[5] R. L Daugherty and J. B. Franzini, *Fluid Mechanics*, 6th ed, McGraw-Hill, New York (1965).